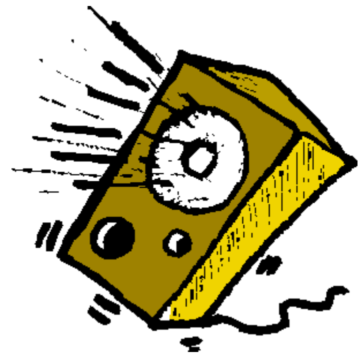
CH 42 – LOG FUNCTIONS

❑ INTRODUCTION



The **decibel scale** for measuring the loudness of sound is discussed at the end of the chapter.

To better understand logs, it would be nice to graph them; but to do that, we need to express the log concept as a function.

You learned how to perform operations like *squaring* back in basic algebra. But it wasn't until later in your studies that you looked at squaring as a function: a collection of inputs and unique outputs where the output is always the square of the input. This new function approach involved concepts such as domain, notation like $f(x) = x^2$, and the parabolic graphs of the squaring functions.

We do the same thing now with *logs*. We've learned what the definition of log is, and we're able to calculate logs, using a variety of bases. Now it's time to formalize the concept of log in the parlance of functions.

❑ DEFINITION

Let b be a positive real number not equal to 1. Then the function

$$y = \log_b x$$

is called the ***log base b*** function.

Note that another way to describe what the base b can be is to say that b must be in the interval $(0, \infty) - \{1\}$.

As with exponential functions, the base, b, of a logarithmic function satisfies the property:

$$b > 0,\ b \neq 1$$

Homework

1. In the log function $y = \log_b x$, what real numbers can b be?

2. Let $f(x) = \log_2 x$. Compute:

   a. $f(2)$ b. $f(4)$ c. $f(16)$ d. $f(128)$ e. $f(1)$

   f. $f(\frac{1}{2})$ g. $f(\frac{1}{8})$ h. $f(0)$ i. $f(-1)$ j. $f(-2)$

3. Let $g(x) = \log x$. Compute:

   a. $g(10)$ b. $g(1)$ c. $g(0)$

   d. $g(100)$ e. $g(-100)$ f. $g(\frac{1}{1000})$

> Note that there is a variety of *log* functions, one for each valid value of the base, *b*.

4. Let $h(x) = \ln x$. Compute:

   a. $h(0)$ b. $h(1)$ c. $h(e)$ d. $h(e^2)$ e. $h(e^{10})$

   f. $h(e^x)$ g. $h(\frac{1}{e})$ h. $h(\sqrt{e})$ i. $h\left(\frac{1}{\sqrt{e}}\right)$ j. $h(-e)$

❑ DOMAIN

From the previous Homework we extract three problems:

$\log_2\left(\frac{1}{8}\right) = -3$ $\log 0$ is Undefined $\ln(-e)$ is Undefined

NOTE: What these 3 logs, along with the rest of the Homework, should tell you is that the log function is defined only for *x*'s that are positive numbers; that is, you can take the log of positive numbers only. Equivalently, you can never take the log of 0 or any negative number. In short,

The <u>domain</u> of the function $y = \log_b x$ is $\boldsymbol{x > 0}$.

[In interval notation, the domain is $(0, \infty)$.]

EXAMPLE 1: **Find the domain of each function:**

A. $f(x) = \log_3(x+7)$

We can take the log of positive quantities only; therefore, we must solve the inequality $x + 7 > 0$, which implies that $x > -7$. So the domain is

$$x > -7$$

Interval notation: $(-7, \infty)$

B. $g(x) = \log(x^2 - 5x - 14)$

Again, we can take log of positive quantities only. So we must solve the inequality $x^2 - 5x - 14 > 0$. Using the Boundary Point Method, we solve the associated equation and calculate the boundary points: $x^2 - 5x - 14 = 0 \Rightarrow (x + 2)(x - 7) = 0 \Rightarrow -2$ and 7 are the boundary points.

By choosing test points and checking them in the original inequality, together with checking the boundary points, you can verify that the domain is

$$x < -2 \text{ OR } x > 7$$

Interval notation:

$\mathbf{(-\infty, -2) \cup (7, \infty)}$

C. $h(x) = \ln(25 - x^2)$

To solve the inequality $25 - x^2 > 0$, we solve the equation to get the boundary points -5 and 5. When the test points and boundary points are tested, the domain is

$$-5 < x < 5$$

Interval notation: $\mathbf{(-5, 5)}$

Homework

5. Consider the function $y = \ln(x^2 - 3x - 10)$. Without actually calculating the domain, determine which of the following are in the domain of the function:

   a. π b. -2 c. -10 d. 0 e. 5 f. e g. 12

6. Find the domain of each function:

   a. $y = \log_{23} x$ b. $y = \log(8 - 2x)$

   c. $y = \ln(x^2 + 2x - 3)$ d. $y = \log_5(x^2 - 49)$

   e. $y = \log_{12}(x^2 + 1)$ f. $y = \log_\pi(100 - x^2)$

❑ GRAPHING

<u>EXAMPLE 2:</u> **Graph:** $\boldsymbol{y = \log_2 x}$

<u>Solution:</u>

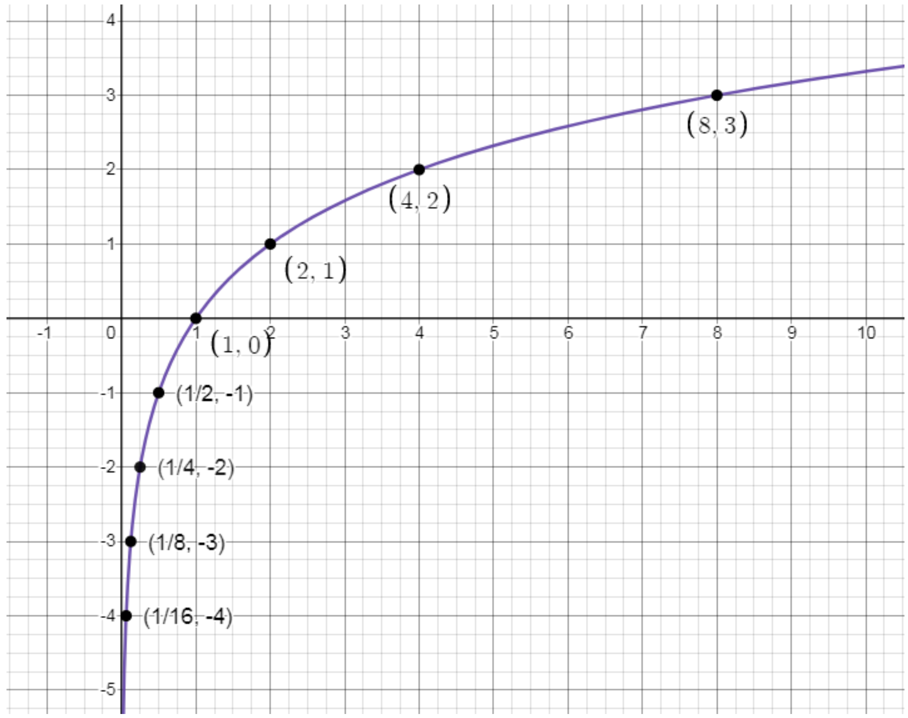
If $x = 8$, then $y = \log_2 8 = 3$ (since $2^3 = 8$).

If $x = 2$, then $y = \log_2 2 = 1$ (since $2^1 = 2$).

If $x = 1/2$, then $y = \log_2 1/2 = -1$ (since $2^{-1} = 1/2$)

Continuing in this way, and remembering that the domain of the log function is $x > 0$, we can calculate the following ordered pairs in the function. Plotting the points in the table and connecting them with a smooth curve gives us the following graph:

x	y
1	0
2	1
4	2
8	3
16	4
$\frac{1}{2}$	-1
$\frac{1}{4}$	-2
$\frac{1}{8}$	-3
$\frac{1}{16}$	-4



Let's confirm what we said earlier. The graph seems to indicate that the **domain** of the function is $\boldsymbol{x > 0}$; that is, x must be a positive real number.

As for **intercepts**, the table and the graph indicate that (1, 0) is the only x-intercept. There is no y-intercept, since x can never be 0.

Let's look at some **limits** for this function. Check out the first five points in the x-y table. As x grows larger and larger, so does y, albeit rather slowly. As slowly as the graph rises, it nevertheless rises higher and higher as x gets larger. Therefore,

$$\text{As } x \to \infty,\ y \to \infty$$

Now look at how the graph gets closer and closer to the y-axis as x gets closer and closer to 0. Remember, even though x can <u>never be</u> 0, that doesn't prevent us from analyzing what happens to y when x <u>approaches</u> 0. But, of course, looking at the graph (and the domain of the function) shows us that x cannot approach 0 from the left because there's no graph there, so we consider x approaching 0 only from the right.

Now to the question: As x approaches 0 from the right, what does y approach? It's going down and down, so it appears to be approaching $-\infty$. Here's what we're saying:

As $x \to 0$ (from the right), $y \to -\infty$

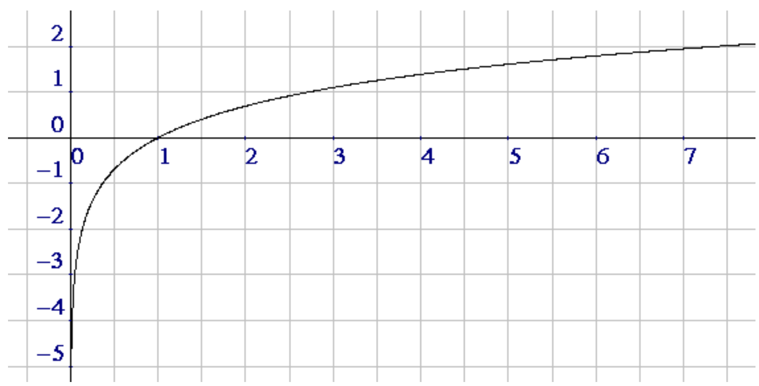
The two limits discussed seem to give evidence that the y-values take on all real numbers. In addition, the second limit indicates that the function has a **vertical asymptote** at $x = 0$ (the y-axis).

EXAMPLE 3: **Graph: $y = \ln x$**

Solution: Using a calculator when necessary, we can compute the following ordered pairs for the function:

$(\frac{1}{10}, -2.3)$	$(\frac{1}{2}, -.69)$	$(1, 0)$
$(e, 1)$	$(5, 1.61)$	$(8, 2.08)$

These points, together with the conclusions of Example 2, yield the following graph:



EXAMPLE 4: **Graph: $y = \log(x + 2)$**

Solution: Note that this is a common log, base 10. We begin our analysis of this function by finding the **domain**. To do this, we create the inequality $x + 2 > 0$, which implies that $x > -2$.

The ***x*-intercept** is found by setting y to 0. This produces the equation log $(x + 2) = 0$. Converting this to log form gives $10^0 = x + 2$, from which it follows that $x = -1$. Thus, the x-intercept is **(–1, 0)**.

To calculate the ***y*-intercept**, we of course set $x = 0$. This yields the equation $y = \log(0 + 2)$, which produces a y-value of about 0.3. Therefore, the y-intercept is **(0, 0.3)**.

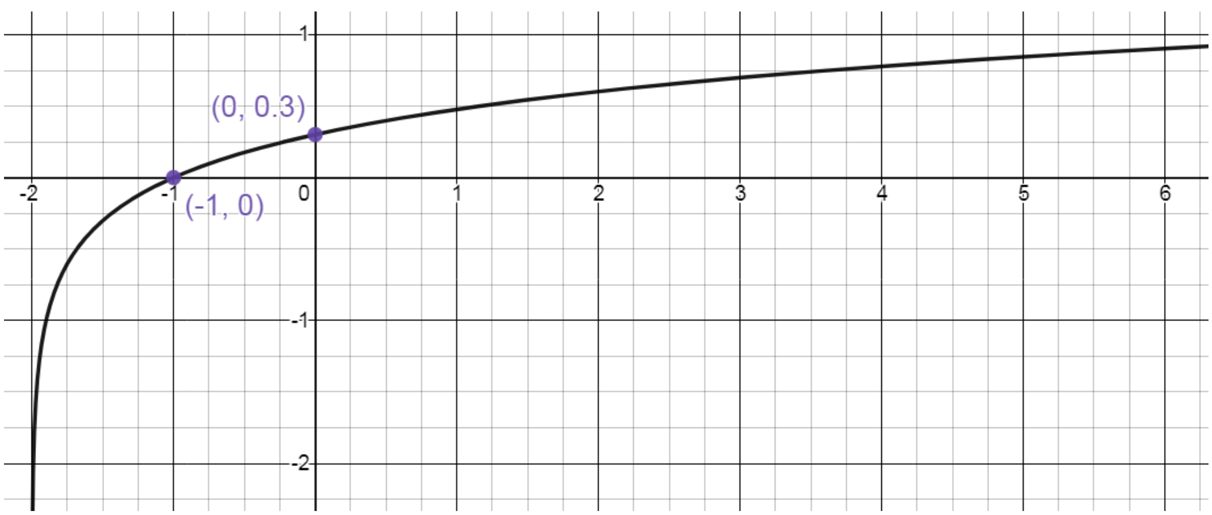
We can find a few ordered pairs for our graph <u>without</u> a calculator by letting $x = -1\frac{9}{10}$, 8, and 98:

$$\left(-1\tfrac{9}{10}, -1\right) \qquad (8, 1) \qquad (98, 2)$$

We can use our calculator to find even more ordered pairs:

$$(-1.98, -1.7) \qquad (3, 0.7) \qquad (5, 0.8) \qquad (6, 0.9)$$

[If you're smart, you'll verify each of these ordered pairs.]



We can be pretty sure that the graph goes infinitely down, and we've already determined that log functions go infinitely up also, although extremely slowly. From these facts, we see that all y-values are obtained by the function. We can summarize these ideas with a pair of limits:

As $x \to -2$ (from the right), $y \to -\infty$

As $x \to \infty$, $y \to \infty$

As a final observation, we can see that the line $x = -2$ is a **vertical asymptote**.

Homework

7. Prove that the graph of $y = \log_2 x$ eventually reaches a height of 10. Hint: Find an x that yields a y of 10.

8. Explain why x cannot approach 0 from the left in the function $f(x) = \log x$.

9. Graph $y = \log(x - 1)$ and describe completely.

10. Graph the three functions $\ln x$, $\ln(x-2)$, and $\ln(x+3)$ on the same grid. Describe how the -2 and the 3 affect the graph of $\ln x$.

11. Find the domain and the vertical asymptote of $y = \ln(x - 5)$.

12. Graph the three functions $\ln x$, $\ln x + 2$, and $\ln x - 1$ on the same grid. Describe how the 2 and -1 affect the graph of $\ln x$. Note: the expression $\ln x + 2$ means take the ln first and then add 2.

13. Find the domain and the vertical asymptote of $y = \ln x + 9$.

❑ LOG FUNCTIONS IN A NUTSHELL

$$y = \log_b x \quad \text{means} \quad x = b^y$$

- y: **any real number**
- b: **positive, $\neq 1$**
- x: **positive**

$$b \in (0, \infty) - \{1\}$$
$$\text{Domain} = (0, \infty)$$

Homework

14. Let $y = \log_b x$. True or false?

    a. x must be a non-negative real number.

    b. y can be any real number.

    c. b could be 1.

    d. The domain of the function is $[0, \infty)$.

    e. y can be any real number.

    f. The equation is equivalent to $b^x = y$.

    g. The graph of the function lies in Quadrants I and II.

    h. x could be 1.

    i. y could be 1.

j. As $x \to \infty$, $y \to 0$.

k. As $x \to -\infty$, $y \to$ nothing at all.

l. As $x \to 0$ (from the left), $y \to -\infty$.

m. As $x \to 1$ (from the right), $y \to 0$.

n. As $x \to 1$ (from the left), $y \to 0$.

o. x must be a positive number.

p. b could be 0.

q. b could be negative.

❑ The Decibel Scale

Named after Alexander Graham Bell, the decibel scale is a measure of the loudness of sound waves. If I is the intensity of the sound, measured in watts per square meter (W/m^2), then the number of decibels is given by

$$D = 10\log\left(\frac{I}{10^{-12}}\right).$$

[The denominator 10^{-12}, or 0.000000000001, is the intensity of the softest sound the human ear can detect.]

Calculator Hint: On a TI-30, to enter a number like 7.3×10^{13}, first press 7.3, then press the "EE" button, and then press 13. To enter the number 10^{-12}, press 1, then EE, then 12, and then +/–.

EXAMPLE 5: **The sound produced by a jet engine has an intensity of 8.3×10^2 W/m^2. Find the decibel value of this sound.**

Solution:

$$\begin{aligned} D &= 10\log\left(\frac{I}{10^{-12}}\right) \\ &= 10\log\left(\frac{8.3\times10^{2}}{10^{-12}}\right) \\ &= 10\log(8.3\times10^{14}) \\ &= 10(14.919) \\ &= 149 \end{aligned}$$

The jet engine sound therefore has a value of

$$\boxed{149 \text{ decibels}}$$

Homework

15. Find the decibel value of a whisper whose intensity is 5.2×10^{-10} W/m^2.

16. Find the decibel value of each sound whose intensity is given:

 a. 1.0×10^{-12} W/m^2
 b. 2.3×10^{-9} W/m^2
 c. 6.3×10^{-7} W/m^2
 d. 8.7×10^{-5} W/m^2

Practice Problems

17. Describe the legal values of b in the function $f(x) = \log_b x$.

18. a. If $f(x) = \log_2 x$, calculate $f(1024) + f(1/8)$.

    b. If $g(x) = \log x$, calculate $g(1000) - g(0.01)$.

    c. If $h(x) = \ln x$, calculate $h(e^6) + h(1) - h(\sqrt[3]{e})$.

    d. Calculate $f(512) + g(1 \text{ million}) + h(e)$

19. Find the domain: $f(x) = \log(35 - 7x)$

20. Find the domain of $y = \ln(x^2 - 144)$.

21. Prove that the function $y = \log_2(x - 2)$ has no y-intercept.

22. If $y = \log_5(7x - 21)$, find x if $y = 2$.

23. Graph: $y = \ln(x - 2)$ State any intercepts and asymptotes.

24. Let $f(x) = \ln x$ and $g(x) = \ln(x - 3) + 4$. Describe the graph of g relative to the graph of f.

25. Consider the function $y = \ln x$. Find the following **limits**:

    a. As $x \to \infty$, $y \to$ _____

    b. As $x \to -\infty$, $y \to$ _____

    c. As $x \to 0$ (from the right), $y \to$ _____

    d. As $x \to 0$ (from the left), $y \to$ _____

26. Use the formula $D = 10\log\left(\frac{I}{10^{-12}}\right)$ to find the decibel level of a sound whose intensity is 7.2×10^{-5} W/m^2.

27. True/False:

a. The base of a log function can be any positive real number.
b. The base of a log function can be any positive real number <u>except 1</u>.
c. The domain of the log function $y = \log_b x$ is $x > 0$.
d. If $g(x) = \log_2(x+10)$, then $g(22) = 5$.
e. The domain of the function $y = \log_6(x^2 - 25)$ is $x \le -5$ OR $x \ge 5$.
f. The x-intercept of the graph of $y = \log(x+2)$ is $(-1, 0)$.
g. The graph of a log function has a horizontal asymptote.
h. The graph of a log function has a vertical asymptote.
i. The graphs of $f(x) = \ln(x+3)$ and $g(x) = \ln x + 3$ are identical.
j. If the intensity of a sound is 2.765×10^3 W/m^2, the decibel value for that sound is about 154.

Solutions

1. b, the base of the log function, can be any real number greater than 0 but not equal to 1. That is, $b > 0$, $b \ne 1$. [This is exactly the same as what b is allowed to be in the exponential function $y = b^x$.]

2. a. 1 b. 2 c. 4 d. 7 e. 0
f. −1 g. −3 h. Undefined i. Undefined j. Undefined

3. a. 1 b. 0 c. Undefined d. 2 e. Undefined f. −3

4. a. Undefined b. 0 c. 1 d. 2 e. 10
f. x g. −1 h. $\frac{1}{2}$ i. $-\frac{1}{2}$ j. Undefined

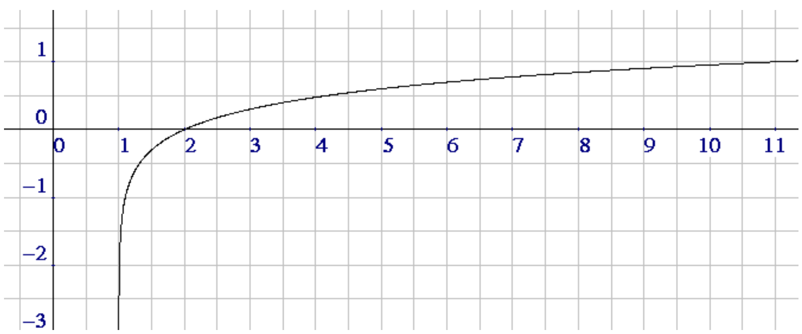
5. c. and g. only, since the quantity $x^2 - 3x - 10$ must be > 0.

6.
a. $x > 0$
b. $8 - 2x > 0 \Rightarrow -2x > -8 \Rightarrow x < 4$
c. $x < -3$ OR $x > 1$
d. $x < -7$ OR $x > 7$
e. $\mathbb{R}$
f. $-10 < x < 10$

7. Setting $10 = \log_2 x \Rightarrow x = 2^{10} = 1024$

8. The domain of $\log x$ is $x > 0$. And x can't approach 0 from the left because x can't be negative; that is, there's no graph to the left of the origin.

9.



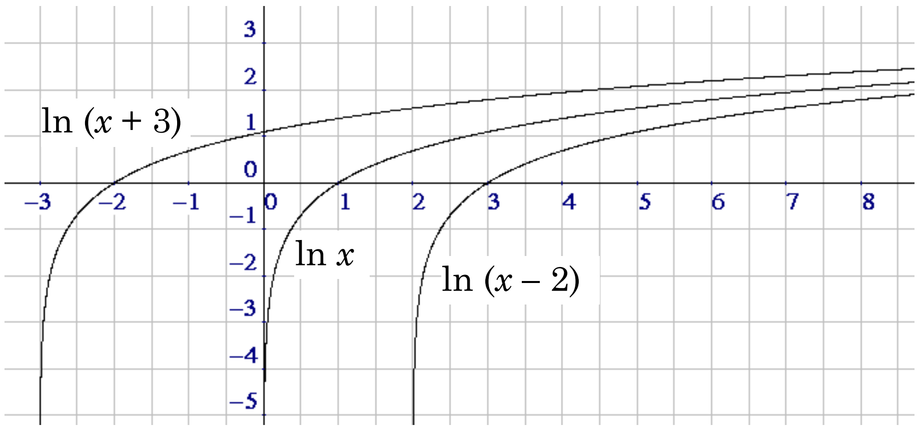
The domain is all real numbers greater than 1: $x > 1$.
The x-intercept is (2, 0).
There is no y-intercept.
There's a vertical asymptote at $x = 1$.
There is no horizontal asymptote.
Some important limits:

The limit as $x \to 1$ (from the left) does not exist.

As $x \to 1$ (from the right), $y \to -\infty$.

As $x \to \infty$, $y \to \infty$.

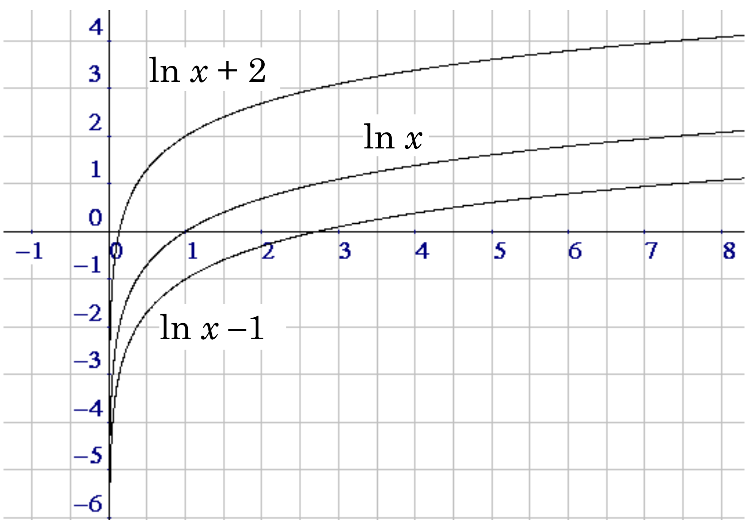
10.



The –2 shifts the graph of $\ln x$ two units to the *right*.
The +3 shifts the graph of $\ln x$ three units to the *left*.

11. Domain = $x > 5$ Vert Asy: $x = 5$

12.



The +2 shifts the graph of $\ln x$ two units *up*.
The –1 shifts the graph of $\ln x$ one unit *down*.

13. Domain = $x > 0$ Vert Asy: $x = 0$

14. a. F b. T c. F d. F e. T f. F g. F h. T i. T
j. F k. T l. F m. T n. T o. T p. F q. F

15. 27 decibels

16. a. 0 b. 33.6 c. 58 d. 79.4

17. $b > 0$, $b \neq 1$

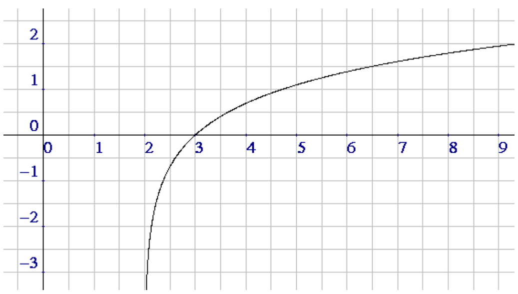
18. a. $10 + (-3) = 7$ b. $3 - (-2) = 5$ c. $6 + 0 - 1/3 = 17/3$
d. $9 + 6 + 1 = 16$

19. $x < 5$ **20**. $x < -12$ OR $x > 12$

21. Setting $x = 0$ yields $\log_2(-2)$, not defined. Therefore, no y-intercept.

22. $2 = \log_5(7x - 21) \Rightarrow 7x - 21 = 5^2 \Rightarrow x = \frac{46}{7}$

23.



x-int: (3, 0)

vert asym: $x = 2$

24. g is obtained by moving f 3 units to the right and 4 units up.

25. a. ∞ b. Does not exist c. $-\infty$ d. Does not exist

26. 78.6 decibels

27. a. F b. T c. T d. T e. F
f. T g. F h. T i. F j. T

Knowledge is the eye of desire and can become the pilot of the soul.

WILL DURANT, AMERICAN WRITER, HISTORIAN, AND PHILOSOPHER